Math 206A Lecture 21 Notes

Daniel Raban

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1 Applications of Dehn's Rigidity Theorem

1.1 Gluck's theorem

Theorem 1.1 (Gluck). *Almost all simplicial polyhedra are rigid.*

By a polyhedron, we mean a 2-dimensional polyhedral surface in $\mathbb{R}^3$ homeomorphic to S^2. The "almost all" is a statement about measure.

Example 1.1. Suppose G is the graph of an octahedron, and let $L : E \to \mathbb{R}_+$ be the edge length function. In the Bricard octahedron, we have restrictions on the lengths of the sides (opposing sides have to have the same length. The idea is that flexible polyhedra are like this; they have 0 measure.

Corollary 1.1 (of Dehn's theorem). *Let $P \subseteq \mathbb{R}^3$ be a simplicial polytope with graph $G = (V, E)$ and length function $L : E \to \mathbb{R}_+$. Then there exists $\varepsilon > 0$ such that for every $L' : E \to \mathbb{R}_+$ with $|L'(e) - L(e)| < \varepsilon$, there exists a convex polytope $P' \subseteq$ combinatorially equivalent to P with length function L'.*

Proof. Let $\mathcal{X}_G$ be the space of all possible length functions $L : E \to \mathbb{R}_+$. Let $n = |V|$. Then $\dim(\mathcal{X}) = 3n - 6$. Let $f_{i,j} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$. We claim that the $f_{i,j}$ are algebraically independent. Let J be the matrix of partial derivatives of $f_{i,j}$. If you look at the partial derivatives, we get that $J = 2R'$, where R' is our rigidity matrix. Then use the inverse function theorem. So there exists an open set around the realization L. $\square$

Here is an incorrect "proof" of Gluck's theorem.

Proof. Let (G, L) be a framework corresponding to the simplicial polyhedron. Take $(i, j) \notin E$. Then $g = f_{i,j} \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n]/\{\text{rigid motions}\}$. Then there exist $c_i \in \mathbb{C}[fe, e \in E]$ such that $c_0 g^N + c_1 g^{N+1} + \cdots + c_N = 0$.

What is the mistake? It is possible that all the coefficients are 0. The idea is that this is a measure zero set. $\square$

There was another issue. We don't know that the a framework for a simplicial polyhedron corresponds to the graph of a simplicial polytope.

Lemma 1.1 (Steinitz theorem for triangulations)**.** *For all $G = (V, E)$ plane triangulations, there exists a convex simplicial polytope $P \subseteq \mathbb{R}^3$ with graph G.*

We will prove this next time.

1.2 Area of polygons given side lengths

Here is an application due to Robbins.

Suppose we have a triangle with side lengths a, b, c. We can find the area of the triangle using Heron's formula. If we have a quadrilateral, we can use another formula (Brahmagupta's formula) to find out the area given the side lengths. Can we do this for pentagons, hexagons, etc.?

How is this related to what we have been talking about? If we have a polygon inscribed in a circle, connect the edges of it via a double suspension (above and below) to a polyhedron (like when we constructed Bricard's octahedron). We can then get a polytope.